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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS

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Abstract. In this paper, we obtain the existence and uniqueness of solution of fractional differential equation with advanced argument. The uniqueness of solution is obtained by using a Banach fixed point theorem with a weighted norm and by a monotone iterative technique, we show the existence of extremal solutions. Examples illustrate the results.

Keywords: *fractional differential equations with advanced arguments, Riemann-Liouville fractional derivatives, existence and uniqueness, monotone iterative technique, extremal solutions.*

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1. Introduction

In [6], Jankowski considered the existence and uniqueness of solutions of the following initial value problem (IVP) for nonlinear Riemann-Liouville fractional differential equations with deviating arguments:

$$\begin{cases} D_{0+}^{q}x(t) = f\big(t, x(t), x(\alpha(t))\big), \ t \in J = [0,T], T > 0, \\ \left[x(t)t^{1-q}\right]\Big|_{t=0} = x_0, \end{cases} \tag{1}$$

where $f \in C(J \times \mathrm{P} \times \mathrm{P},\ \mathrm{P})$, $\alpha \in C(J, J)$, $\alpha(t) \le t,\ t \in J$ and $0 < q < 1$, by using the Banach fixed point theorem and monotone iterative method.

In this paper, we investigate the following IVP for nonlinear fractional differential equation with advanced argument:

$$\begin{cases} D_{0+}^{\alpha}x(t) = f\big(t, x(t), x(\theta(t))\big),\ t \in J = [0,T],\ T > 0, \\ \quad \tilde{x}(0) = g(x), \end{cases} \tag{2}$$

where $f\big(t, x(t), x(\theta(t))\big) \in C\big([0,T] \times \mathrm{P}^2, \mathrm{P}\big), \theta \in C(J, J), t \le \theta(t) \le T,\ t \in J$, $g : C_{1-\alpha}(J) \to \mathrm{P}$ is a continuous functional, $\tilde{x}(0) = t^{1-\alpha}x(t)\big|_{t=0}$ and $D_{0+}^{\alpha}x(t)$ is the Riemann-Liouville fractional derivative of x of order α $(0 < \alpha < 1)$.

Since $f\big(t, x(t), x(\theta(t))\big)$ is continuous, the nonlinear IVP (2) is equivalent to the following integral equation

$$x(t) = g(x)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, x(s), x(\theta(s))\right)ds, \tag{3}$$

where Γ denotes the gamma function. Advanced differential equations are very important in industry, engineering, economics and so on [4, 5]. Firstly, uniqueness of solution is obtained by using a Banach fixed point theorem with weighted norm, can be found in [1, 3, 6, 8, 10, 17, 19]. However, discussion on initial value problems of fractional differential equations with advanced arguments is rare. Secondly, in [6], in order to discuss the existence and uniqueness of IVP(1), Jankowski divided $0 < q < 1$ into two situations, one is $0 < q \le 1/2$ with an additional condition and the other is $1/2 < q < 1$. In this paper, we unify the two situations without using the additional condition. Thirdly, for the study of fractional differential equation, existence of extremal solutions of problem [6] is a useful tool (see [9, 11, 12, 13, 19]). We know that it is important to build a comparison result when we use the monotone iterative technique. It makes the calculation easier and is suitable for the more complicated forms of equations.

The paper is organized as follows: In Section 2, we present some useful definitions and fundamental facts of fractional calculus. In Section 3, by applying Banach fixed point theorem with the corresponding weighted norm, we prove the uniqueness of solution for nonlinear IVP (2). In Section 4, we develop the monotone iterative technique and apply it to prove that existence of extremal solutions of nonlinear IVP (2). Lastly, we illustrate our results with suitable examples.

2. Preliminaries

For the reader's convenience, we present some necessary definitions from fractional calculus and Lemma. Let $C_{1-\alpha}(J,\mathrm{P}) = \{x \in C((0,T],\mathrm{P}):\ t^{1-\alpha}x \in C(J,\mathrm{P})\}$ with the norm

$$\|x\|_{C_{1-\alpha}} = \max_{t\in[0,T]}\left|t^{1-\alpha}x(t)\right|,$$

where λ is a fixed positive constant. Clearly, the space $C_{1-\alpha}(J,\mathrm{P})$ is a Banach space. Now, let us recall the following definitions from fractional calculus (for details see [7, 14]).

Definition 1. For $\alpha > 0$, the integral

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds$$

is called the Riemann-Liouville fractional integral of order α, where Γ is the gamma function. Note that, $I_{0+}^{\alpha} f(t)$ is defined on $[0,T]$ for $f \in L_1(0,T)$.

Definition 2. For function $I_{0+}^{n-\alpha} f(t) \in AC^n[0,T]$ the Riemann-Liouville fractional derivative of order α $(n-1 < \alpha \le n)$ can be written as

$$D_{0+}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n \left(I_{0+}^{n-\alpha} f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,\ t > 0.$$

Lemma 1 [7]. Let $n-1<\alpha\le n$ and let $f_{n-\alpha}(t)=I_{0+}^{n-\alpha}f(t)$ be fractional integral of order $n-\alpha$. If $f(t)\in L(0,T)$ and $f_{n-\alpha}(t)\in AC^n[0,T]$, then we have the following equality

$$I_{0+}^{\alpha}D_{0+}^{\alpha}f(t)=f(t)-\sum_{j=1}^{n}\frac{f_{n-\alpha}^{(n-j)}(0)}{\Gamma(\alpha-j+1)}t^{\alpha-j},\ f_{n-\alpha}(t)=I_{0+}^{n-\alpha}f(t). \qquad (4)$$

3. Uniqueness of Solution

In this section, we discuss the uniqueness of solution for nonlinear IVP (2) for Riemann-Liouville fractional differential equation with advanced argument under the following conditions:

(H_1) There exist nonnegative constants L_1, L_2 such that

$$\left|f(t,v_1,v_2)-f(t,u_1,u_2)\right|\le L_1|v_1-u_1|+L_2|v_2-u_2|,\ \forall t\in J,\ v_i,u_i\in \mathrm{P},\ i=1,2,$$

(H_2) There exists a constant $L_3\in(0,1)$ such that

$$|g(u_1)-g(u_2)|\le L_3\|u_1-u_2\|_{C_{1-\alpha}},\ \forall t\in J,\ \forall u_1,u_2\in C_{1-\alpha}(J).$$

Lemma 2. Let $f\left(t,x(t),x(\theta(t))\right)\in C\left(J\times \mathrm{P}^2,\mathrm{P}\right)$. Suppose $x(t)$ is a solution of nonlinear IVP (2), if and only if $x(t)$ is a solution of integral equation

$$x(t)=g(x)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,x(s),x(\theta(s))\right)ds\,. \qquad (5)$$

Proof. Assume that $x(t)$ satisfies IVP (2). From the first equation of IVP (2) and Lemma 1, we have

$$x(t)=\frac{I_{0+}^{1-\alpha}x(t)\big|_{t=0}t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,x(s),x(\theta(s))\right)ds$$

$$=g(x)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,x(s),x(\theta(s))\right)ds.$$

Conversely, assume that $x(t)$ satisfies (5). It is easy to check $x(t)\in C_{1-\alpha}(J)$. Applying the operator D_{0+}^{α} to both sides of (5), we have

$$D_{0+}^{\alpha}x(t)=f\left(t,x(t),x(\theta(t))\right).$$

In addition, we have $\tilde{x}(0)=t^{1-\alpha}x(t)\big|_{t=0}=g(x)$. The proof is complete.

Theorem 1. Let (H_1)-(H_2) hold, $f\in C(J\times \mathrm{P}^2,\mathrm{P})$. Then the nonlinear IVP (2) has a unique solution.

Proof. Define the operator $T: C_{1-\alpha}(J)\to C_{1-\alpha}(J)$ by

$$Tx(t)=g(x)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,x(s),x(\theta(s))\right)ds. \qquad (6)$$

Clearly, the operator T is well defined on $C_{1-\alpha}(J)$. Next, we show that T is a contraction operator $C_{1-\alpha}(J)$. For convenience, let

$$\left\{L_3+\frac{\Gamma(\alpha)T^\alpha}{\Gamma(2\alpha)}\left(L_1+L_2\right)\right\}<1. \tag{7}$$

Using assumption $(H_1),(H_2)$. For any $x, y \in C_{1-\alpha}(J)$, we have

$$\begin{aligned}
\|Tx-Ty\|_{C_{1-\alpha}} &= \max_{t\in J}\left|t^{1-\alpha}\left[(Tx)(t)-(Ty)(t)\right]\right| \\
&\le |g(x)-g(y)|+\max_{t\in J}\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,x(s),x(\theta(s)))-f(s,y(s),y(\theta(s)))\right|ds \\
&\le L_3\|x-y\|_{C_{1-\alpha}}+\tfrac{1}{\Gamma(\alpha)}\max_{t\in J} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\left[L_1|x(s)-y(s)|+L_2|x(\theta(s))-y(\theta(s))|\right]ds \\
&\le L_3\|x-y\|_{C_{1-\alpha}}+\frac{1}{\Gamma(\alpha)}\max_{t\in J} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\left[L_1 s^{\alpha-1}\|x-y\|_{C_{1-\alpha}}+L_2(\theta(s))^{\alpha-1}\|x-y\|_{C_{1-\alpha}}\right]ds \\
&\le L_3\|x-y\|_{C_{1-\alpha}}+\frac{\left(L_1+L_2\right)}{\Gamma(\alpha)}\max_{t\in J} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds\|x-y\|_{C_{1-\alpha}} \\
&\le L_3\|x-y\|_{C_{1-\alpha}}+\max_{t\in J}\left(\frac{L_1+L_2}{\Gamma(\alpha)}\right)t^\alpha\int_0^1 (1-\eta)^{\alpha-1}\eta^{\alpha-1}d\eta\|x-y\|_{C_{1-\alpha}} \\
&\le \left\{L_3+\frac{\Gamma(\alpha)T^\alpha}{\Gamma(2\alpha)}\left(L_1+L_2\right)\right\}\|x-y\|_{C_{1-\alpha}}
\end{aligned}$$

According to (7) and using Banach fixed point theorem, the nonlinear IVP (2) has a unique solution. The proof is complete.

Lemma 3. Let M, $N \in C(J)$, and $|M(t)| \le M_1$, $|N(t)| \le N_1$, $\forall t \in J$, $\sigma \in C_{1-\alpha}(J)$. The linear initial value problem:

$$\begin{cases} D_{0+}^\alpha x(t) = M(t)x(t)+N(t)x(\theta(t))+\sigma(t),\ t\in[0,T],\ 0<\alpha<1, \\ \qquad \tilde{x}(0)=t^{1-\alpha}x(t)\big|_{t=0}=g(x) \end{cases} \tag{8}$$

has a unique solution

$$x(t)=g(x)t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(s)x(s)+N(s)x(\theta(s))+\sigma(s)\right]ds. \tag{9}$$

Proof. It follows from Theorem 1.

Remark 1.

i. Putting $M(t)=0$ and $N(t)=0$, in the above linear IVP (8), we have the results obtained by Zhang [19];
ii. Putting $N(t)=0$, in the above the linear IVP (8), we have the results obtained by Kilbas et al., [7], Zhang [19], Wei et al., [16];
iii. Putting $\alpha=1$ and $N(t)=0$, in the above linear IVP (8) is $x'(t)=Mx(t)+\sigma(t)$, $\tilde{x}(0)=g(x)$, we have the results obtained by Bonilla, [2], Kilbas et al., [7], Wei et al., [16];
iv. Putting $N(t)=0$ and $\sigma(t)\equiv 0$, in the above linear IVP (8), we get the solution of the corresponding homogeneous IVP (8) of the form $x(t)=\Gamma(\alpha)g(x)t^{\alpha-1}E_{\alpha,\alpha}(Mt^\alpha),\ t\in[0,T]$ (see Kilbas et al., [7]).

4. The Monotone Iterative Technique

In this section, the monotone iterative technique for the nonlinear IVP (2) of Riemann Liouville fractional differential equation with advanced argument is developed and obtained the existence of extremal solutions of IVP (2). We need the following comparison result which will play a very important role in our further discussion.

Lemma 4. Let $0<\alpha<1$, M, $N \in C(J)$, $|M(t)| \le M_1$, $|N(t)| \le N_1$.

Suppose that

$$\frac{T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\left(M_1+N_1\right)<1 \tag{10}$$

holds, and $p \in C_{1-\alpha}(J)$ satisfies

$$p(t) \le \tilde{p}(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(s)p(s)+N(s)p(\theta(s))\right]ds,$$

$$\tilde{p}(0) \le 0.$$

Then $p(t) \le 0$ for all $t \in J$.

Proof. It is by method of contradiction. Suppose that $p(t) \not\le 0$, $\forall t \in J$. So there exists at least one $t_* \in J$ such that $t_*^{1-\alpha}p(t_*)>0$. Without loss of generality, we assume

$$t_*^{1-\alpha}p(t_*) = \max_{t\in[0,T]}\left\{t^{1-\alpha}p(t)\right\} = \rho_1 > 0.$$

We obtain that

$$\begin{aligned} t^{1-\alpha}p(t) &\le \tilde{p}(0) + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(s)p(s)+N(s)p(\theta(s))\right]ds \\ &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(s)p(s)+N(s)p(\theta(s))\right]ds \\ &\le \frac{M_1t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}|p(s)|ds \\ &\quad + \frac{N_1t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\theta(s))^{\alpha-1}(\theta(s))^{1-\alpha}|p(\theta(s))|ds \end{aligned}$$

Let $t=t_*$, we have

$$\rho_1 \le \left(\frac{T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\left(M_1+N_1\right)\right)\rho_1 .$$

So

$$\frac{T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\left(M_1+N_1\right)\ge 1,$$

which contradicts inequality (10). Hence $p(t) \le 0$ for all $t \in J$. The proof is complete.

Remark 2. If $M(t) \ge 0$ and $N(t) = 0$ on J, then Lemma 4 reduces to (Wang, [15], Lemma 2.5) (note that condition $\frac{T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)} M_1(t) < 1$ from (Wang, [15]) is superfluous).

Definition 3. A function $x_0 \in C_{1-\alpha}(J)$ is called lower solution of IVP (2) if

$$\begin{cases} x_0(t) \le \tilde{x}_0(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s, x_0(s), x_0(\theta(s))\big)ds, \ \ t \in J \\ \tilde{x}_0(0) \le g(x_0). \end{cases}$$

Analogously, $y_0 \in C_{1-\alpha}(J)$ is called an upper solution of nonlinear IVP (2) if

$$\begin{cases} y_0(t) \ge \tilde{y}_0(0)t^{\alpha-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{\alpha-1} f\big(s, y_0(s), y_0(\theta(s))\big)ds, \ t \in J \\ \tilde{y}_0(0) \ge g(y_0). \end{cases}$$

In further discussion, we need the following assumptions:

(H_3) Functions x_0 and y_0 are ordered lower and upper solutions of nonlinear IVP (2) such that $x_0(t) \le y_0(t), \ t \in J.$

(H_4) There exist M, $N \in C(J)$ such that

$$f(t,x,y) - f(t,\bar{x},\bar{y}) \ge M(t)(x-\bar{x}) + N(t)(y-\bar{y})$$

where $x_0(t) \le \bar{x}(t) \le x(t) \le y_0(t), \ x_0(\theta(t)) \le \bar{y}(\theta(t) \le y(\theta(t) \le y_0(\theta(t)).$

(H_5) The function g satisfies $g(x) - g(\bar{x}) \ge x - \bar{x}$, for $\tilde{x}_0(0) \le \bar{x} \le x \le \tilde{y}_0(0).$

Theorem 2. Let (H_3)-(H_5) and inequality (10) hold. Then there exist two monotone sequences $\{x_n\},\{y_n\} \subset [x_0, y_0]$ which converge uniformly to the extremal solutions of nonlinear IVP (2) in the sector $[x_0, y_0]$, where $[x_0, y_0] = \{z \in C_{1-\alpha}(J): \ x_0(t) \le z(t) \le y_0(t), \ t \in J, \ \tilde{x}_0(0) \le \tilde{z}(0) \le \tilde{y}_0(0)\}.$

Proof. The proof consists of the following three steps.

Step 1. Construct two the sequences $\{x_n\},\{y_n\}$. For any $\eta \in [x_0, y_0]$, we consider the following linear initial value problem:

$$\begin{cases} D_{0+}^{\alpha} x(t) = M(t)x(t) + N(t)x(\theta(t)) + \sigma(t) \\ \quad \tilde{x}(0) = g(\eta), \end{cases} \tag{11}$$

where $\sigma(t) = f\big(t, \eta(t), \eta(\theta(t))\big) - M(t)\eta(t) - N(t)\eta(\theta(t)).$

Obviously, by Lemma 3, the linear IVP (11) has a unique solution which satisfies

$$\begin{cases} x(t) = \tilde{x}(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[M(s)x(s) + N(s)x(\theta(s)) + \sigma(s)\big]ds, \\ \tilde{x}(0) = g(\eta). \end{cases} \tag{12}$$

Suppose that $x_1(t)$ and $x_2(t)$ are two solutions of linear IVP (11). Let $p(t) = x_1(t) - x_2(t)$. Applying Lemma 4 again one can prove that $p(t) \le 0$, and thus $x_1(t) \le x_2(t)$. As the same argument is valid for $x_2(t) - x_1(t)$, we conclude that $x_1(t) = x_2(t)$. This proves uniqueness.

Now, we define an operator $A : [x_0, y_0] \to [x_0, y_0]$ by $x = A\eta$. Clearly, the operator A is well defined on $[x_0, y_0]$, let $\eta_1, \eta_2 \in [x_0, y_0]$ such that $\eta_1 \le \eta_2$. Suppose that $z_1(t) = A\eta_1(t)$ and $z_2(t) = A\eta_2(t)$. Setting $p(t) = z_1(t) - z_2(t)$, we obtain that

$$p(t) \le \tilde{p}(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(s)p(s) + N(s)p(\theta(s))\right]ds$$

and

$$\tilde{p}(0) = \tilde{z}_1(0) - \tilde{z}_2(0) = g(\eta_1) - g(\eta_2) \le 0.$$

By Lemma 4, we get $p(t) \le 0$, implies $A\eta_1(t) \le A\eta_2(t)$ for all $t \in J$. It means that A is nondecreasing. Obviously, we can easily get that A is a continuous map. Let $x_n = Ax_{n-1},\ y_n = Ay_{n-1},\ n = 1, 2, \ldots$.

Step 2. The sequences $\{x_n\}$ and $\{y_n\}$ converge uniformly to x^*, y^* respectively. In fact $\{x_n\}$ and $\{y_n\}$ satisfy the following relation

$$x_0 \le x_1 \le \cdots \le x_n \le \cdots x^* \le y^* \cdots \le y_n \le \cdots \le y_1 \le y_0. \tag{13}$$

Setting $p(t) = x_0(t) - x_1(t)$ and $x_0(t)$ is the lower solution of nonlinear IVP (2), we obtain

$$p(t) \le \tilde{p}(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(s)p(s) + N(s)p(\theta(s))\right]ds$$

and

$$\tilde{p}(0) = \tilde{x}_0(0) - \tilde{x}_1(0) \le g(x_0) - g(x_0) = 0.$$

By Lemma 4, we get $p(t) \le 0$, implies $x_0(t) \le x_1(t)$ for all $t \in J$. Similarly, we can show that $y_1 \le y_0$ for all $t \in J$. Applying the operator A to both sides of $x_0(t) \le x_1(t)$, $y_1 \le y_0$ and $x_1 \le x_0$, we can easily get (13). Obviously, the sequences $\{x_n\}$ and $\{y_n\}$ are uniformly bounded and equicontinuous. Hence by the Ascoli-Arzela Theorem the sequences $\{x_n\}$ and $\{y_n\}$ converge uniformly on J with

$$\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*.$$

Step 3. Prove that x^*, y^* are extremal solutions of nonlinear IVP (2), and x^*, y^* are solutions of nonlinear IVP (2) on $[x_0, y_0]$, because of the continuity of operator A. Let $z \in [x_0, y_0]$ be any solution of nonlinear IVP (2). That is,

$$\begin{cases} z(t) = \tilde{z}(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, z(s), z(\theta(s))\right)ds, \\ \tilde{z}(0) = g(z). \end{cases}$$

Suppose that there exists a positive integer n such that $x_n(t) \le z(t) \le y_n(t)$ on J. Let $p(t) = x_{n+1}(t) - z(t)$.

Then we have

$$p(t) \le \tilde{p}(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[M(s)p(s) + N(s)p(\theta(s))]ds.$$

and

$$\tilde{p}(0) = \tilde{x}_{n+1}(0) - \tilde{z}(0) = g(x_n) - g(z) \le 0.$$

By Lemma 4, we get $p(t) \le 0$, implies $x_{n+1}(t) \le z(t)$ for all $t \in J$. Similarly, we can get $z(t) \le y_{n+1}(t)$ for all $t \in J$. Since $x_0(t) \le z(t) \le y_0(t)$ for all $t \in J$, by mathematical induction, we get that $x_n(t) \le z(t) \le y_n(t)$ on J for all n. Therefore, $x^*(t) \le z(t) \le y^*(t)$ on J by taking $n \to \infty$. The proof is complete.

5. Examples

Example 1. Consider the following IVP

$$\begin{cases} D_{0+}^{\frac{1}{2}} x(t) = t + \frac{1}{60}x(t) + \frac{1}{30}x(\sqrt{t}),\ t \in [0,1], \\ \tilde{x}(0) = g(x) = \frac{\sqrt{\eta}}{8}x(\eta),\ 0 < \eta < 1, \end{cases} \tag{14}$$

where $\alpha = \frac{1}{2}$, $T = 1$, $\theta(t) = \sqrt{t}$ and $f(t, x(t), x(\sqrt{t})) = t + \frac{1}{60}x(t) + \frac{1}{30}x(\sqrt{t})$. Obviously, $f(t, x(t), x(\sqrt{t}))$ satisfies Lipschitz condition and there exist nonnegative constants $L_1 = \frac{1}{60}$, $L_2 = \frac{1}{30}$ such that

$$\left| f(t, x(t), x(\sqrt{t})) - f(t, y(t), y(\sqrt{t})) \right| \le \frac{1}{60}|x(t) - y(t)| + \frac{1}{30}\left|x(\sqrt{t}) - y(\sqrt{t})\right| \text{ if } t \in J,$$

which shows that condition (H_1) of Theorem 1 holds.

Also, there exists constant $L_3 = \frac{1}{8} \in (0,1)$. Moreover,

$$|g(x) - g(y)| \le \frac{1}{8}\|x - y\|_{C_{\frac{1}{2}}}.$$

So the condition (H_2) of Theorem 1 is satisfied. The IVP (14) has a unique solution. Consider the same equation as in IVP (14), taking $x_0(t) = 0$, $y_0(t) = 10t^{-\frac{1}{2}} + 6$, and then we have $\tilde{y}_0(0) = 10$. Moreover,

$$y_0(t) = 10t^{-\frac{1}{2}} + 6 \ge 10t^{-\frac{1}{2}} + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}\left(s + \frac{1}{60}\left(10s^{-\frac{1}{2}} + 6\right) + \frac{1}{30}\left(10s^{-\frac{1}{2}} + 6\right)^{\frac{1}{2}} \right)ds,$$

$$\tilde{y}_0(0) = 10 \ge \frac{\sqrt{\eta}}{8}[10\eta^{-\frac{1}{2}} + 6],\ 0 < \eta < 1.$$

On the other hand, it is easy to check that $x_0 \le y_0$ and (H_3) holds.

And let $M(t) = \dfrac{t^2}{60}$, $N(t) = \sin\dfrac{t}{30}$, we get that

$$f(t,x(t),x(\sqrt{t}))-f(t,y(t),y(\sqrt{t}))\geq\frac{t^2}{60}\left[x(t)-y(t)\right]+\sin\frac{t}{30}\left[x(\sqrt{t})-y(\sqrt{t})\right],$$

where $x_0\leq u_1\leq v_1\leq y_0$, $x_0(\sqrt{t})\leq u_2\leq v_2\leq y_0(\sqrt{t})$. So (H_4) is satisfied.

We see that $M_1=\dfrac{1}{60}$, $N_1=\dfrac{1}{30}$, which implies

$$\frac{T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\left(M_1+N_1\right)=\frac{\pi^{\frac{1}{2}}}{20}<1.$$

It is easy to see that inequality (10) holds. Thus, all conditions of Theorem 2 are satisfied. Therefore, the IVP (14) has extremal solutions.

Example 2. Consider the following IVP

$$\begin{cases} D_{0+}^{\frac{1}{2}}x(t)=\frac{t^2}{30}x(t)+\frac{t^3}{15}x(2t),\ t\in[0,1] \\ \tilde{x}(0)=g(x)=\frac{\sqrt{\eta}}{12}x(\eta),\ 0<\eta<1. \end{cases} \tag{15}$$

Obviously, $\alpha=\frac{1}{2}$, $\theta(t)=2t$, $T=1$ and $f(t,x(t),x(2t))=\frac{t^2}{30}x(t)+\frac{t^3}{15}x(2t)$.

Let $L_1=\frac{1}{30}$, $L_2=\frac{1}{15}$ and $L_3=\frac{1}{12}$. It is easy to check that

$$\left|f(t,x(t),x(2t))-f(t,y(t),y(2t))\right|\leq\frac{1}{30}\left|x(t)-y(t)\right|+\frac{1}{15}\left|x(2t)-y(2t)\right|,$$

$$\left|g(x)-g(y)\right|\leq\frac{1}{12}\left\|x-y\right\|_{C_{\frac{1}{2}}}.$$

So, (H_1) and (H_2) of Theorem1 hold. Thus, all conditions of Theorem 1 are satisfied, the IVP (15) has a unique solution.

Consider the same equation as in IVP (15), taking $x_0(t)=0$, $y_0(t)=t^{-\frac{1}{2}}+6$, and then we have $\tilde{y}_0(0)=1$. Moreover,

$$y_0(t)=t^{-\frac{1}{2}}+6\geq t^{-\frac{1}{2}}+\frac{1}{\Gamma(\frac{1}{2})}\int_0^t(t-s)^{-\frac{1}{2}}\left(\frac{s^2}{30}\left(s^{-\frac{1}{2}}+6\right)+\frac{2s^3}{15}\left(s^{-\frac{1}{2}}+6\right)\right)ds,$$

$$\tilde{y}_0(0)=1\geq\frac{1}{12}+\frac{\sqrt{\eta}}{2},\ 0<\eta<1.$$

On the other hand, it is easy to check that $x_0\leq y_0$ and (H_3) holds. And let $M(t)=\frac{t^2}{60}$, $N(t)=\frac{\sin te^{-2e\sqrt{t}}}{30}$, we get that

$$f(t,x(t),x(2t))-f(t,y(t),y(2t))\geq\frac{t^2}{60}\left[x(t)-y(t)\right]+\frac{\sin te^{-2e\sqrt{t}}}{30}\left[x(2t)-y(2t)\right]$$

where $x_0\leq y\leq x\leq y_0$, $x_0(2t)\leq y(2t)\leq x(2t)\leq y_0(2t)$. So (H_4) is satisfied.

Obviously, $M_1=\frac{1}{30}$, $N_1=\frac{1}{15}$, and then we can get

$$\frac{T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\left[M_1+N_1\right]=\frac{\pi^{\frac{1}{2}}}{10}<1.$$

Inequality (10) holds. All conditions of Theorem 2 are satisfied, so IVP (15) has extremal solutions.

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